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BUCKLING AND VIBRATION OF MULTILAYER ORTHOTROPIC COMPOSITE SHELLS USING A SIMPLE HIGHER-ORDER LAYERWISE THEORY

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Abstract The buckling and vibration of thick, orthotropic laminated composite shells is modelled using a simple layerwise higher-order theory. The theory accounts for a cubic variation of both the in-plane displacements and the transverse shear stresses within each layer, the latter being zero at the free surfaces without the need for shear correction factors. By imposing the continuity of the in-plane displacements and the transverse shear stresses at the interfaces, the number of variables is shown to be the same as that given by the FSDT, irrespective of the number of layers considered. A non-dimensionalized parameter called the General Performance Index is defined in order to assess the overall performance of the models based on their flexural frequencies and the largest component of stress within the laminate. Numerical results for moderately short, one-, two- and three-layer shell panels are obtained for a range of base layer-to-core modulus of elasticity ratios. The normalized natural frequencies and stresses of the present theory are compared with a simple layerwise first-order theory and two other global higher-order theories that despite their similarity, indicate some interesting differences. The critical buckling loads are also given for a range of modulus and thickness ratios. Results indicate the present theory generally performs better over a range of the parameters mentioned predicting conservatively lower natural frequencies, smaller buckling loads and larger stresses for symmetric shells.

INTRODUCTION

The Classical Theory of Shells (CST) was proposed by Love (1888) for the deformation and oscillation of shells with the assumptions that the shell is thin, deflections are small, transverse normal stress is negligible and that normals to the reference surface remain normal to it without any change in length during motion. Four decades later, Donnell (1933) developed a set of equations from the CST for cylindrical shells by making the simplifying assumptions that the transverse shear force Q_s is of a much smaller order than other forces in the circumferential direction, and that the in-plane displacement u_s in the same direction has negligible effect on the curvature and twist of the shell. These modifications were tested for their range of validity by Hoff (1955) using the theory of Flügge (1932) as a benchmark. The CST was not used in this verification owing to the contradictions associated with the equality of the shearing forces N_{12} and N_{21} as well the shearing moments M_{12} and M_{21}. These contradictions do not appear for spherical shells and circular plates due to the nature of their geometries. However, since Donnell's (1933) equations were not derived for any of these cases, the more consistent Flügge (1932) equations for cylindrical shells were used. For purposes of making easy comparisons, the original Flügge (1932) equations were recast by Kempner (1955) into a form similar to that of Donnell (1933). In doing so, all terms of the order of $(h/R)^2$ and higher were neglected in the Flügge–Kempner equations in order to make a credible comparison with the Donnell (1933) equivalent. The results of Hoff (1955) showed that the Donnell (1933) equations agreed satisfactorily with the Flügge–Kempner equations only for short, closed shells and for higher modes in the circumferential direction. Similar results for the vibration of cross-ply cylindrical shells was obtained by Soldatos (1984) in which Donnell's (1933) equations overestimated the flexural frequencies when compared to results of Love (1952), Flügge (1973) and Sanders (1959) for long, shallow panels. Morely (1959) proposed a set of equations that improved Donnell's (1933) first approximation especially for long cylinders under edge or distributed loading.

While these simplifications and improvements on the CST were taking place, other researchers, notably Epstein (1942) chose to abandon energy considerations and use the equations of equilibrium from Newton's laws. Kennard (1953) made use of Epstein's (1942) fundamental work to develop distributions of displacements and stresses. In this treatment, all derivatives of the displacements and stresses with respect to the radial direction were eliminated by expressing these quantities as a Taylor series in the thickness coordinate. The unknown variables in these series are functions of the in-plane orthogonal coordinates only, as they pertain to the reference surface. Thus the original differential equations were converted to a set of algebraic equations. In what may be considered to be a complete analysis, terms up to the second order of the series for displacements and stresses were included involving some 12 equations and 12 unknown variables. Computationwise, it would be a rather cumbersome if terms of the cubic order were added to improve accuracy as an additional six unknown variables would be introduced requiring Kennard's (1953) equations (9a–c) and (10a–c), to be differentiated once again with respect to the thickness coordinate in order to give six more equations which include zero-order values of the additional unknown variables.

A period followed in which first-order theories accounting for transverse shear deformation and rotary inertia were popular among researchers. Mirsky and Herrmann (1956) proposed such a first-order theory for the nonaxially symmetric motions of thin cylindrical shells. In the study of torsionless axisymmetric wave propagation in an infinite cylindrical shell, Naghdi and Cooper (1956) independantly derived two general systems of equations of motion one of which is similar to that obtained by Mirsky and Herrmann (1956). The two systems of equations are such that, one of these reduces to the set by Love (1888) and the other to that given by Donnell (1933) when the transverse shear deformation and rotary inertia terms were removed.

With the advancement in technology, came the use of new materials of superior strength whose properties were directional. This led to investigations by Mirsky (1964) into the effects of transverse shear deformation, rotary inertia and thickness deformation on the vibrations of thick, orthotropic shells. In order to assess the accuracy of this model, comparisons were made with exact elasticity results for isotropic materials. It was found that solutions to the first four modes compared well and, therefore, it was assumed that solutions for the orthotropic case would give the same order of accuracy.

In order to achieve the desired strength and stiffness in certain directions, composite materials were laminated and stacked in chosen sequences. Dong and Tso (1972) derived a first-order transverse shear deformation theory for such laminated orthotropic shells complete with shear correction factors. Di Sciuva (1987) extended the layerwise theory of Waltz and Vinson (1976) to multilayered anisotropic composite shells in order to account for realistic continuity conditions at the interfaces. This multilayered anisotropic shell theory has been applied by Di Sciuva and Carrera (1992) to the elastodynamic behaviour of thick, symmetrically laminated, anisotropic circular cylindrical shells. Numerical results for simply-supported circular cylindrical shells were found to compare well with those obtained by Reddy and Liu (1985), irrespective of radius-to-length and length-to-thickness ratios.

Sun and Whitney (1974) were among the few researchers who included higher-order terms in modelling vibrations of laminated composite cylindrical shells. The in-plane displacement fields remained linear while the distribution in the transverse direction was quadratic, giving rise to certain additional modes that appeared only at higher frequencies. Results from this theory when compared with Flugge's (1967), showed that at progressively large thickness to length ratios, the discrepancy varied rapidly due to the significantly increasing transverse shear deformation. For the first mode, differences were observed to be more significant for four-layered, angle-ply laminates than for the corresponding symmetric cross-ply ones. Bhimaraddi (1984) proposed a higher-order theory in which in-plane displacements were cubic in distribution while the transverse deflection was invariant with thickness. This resulted in parabolic variation of transverse shear stresses with zero values at the free surfaces without the need for shear correction factors. In a separate development, Reddy and Liu (1985) derived a similar model in which all terms of the order thickness coordinate to radius were neglected with the exception of those appearing as the coefficient

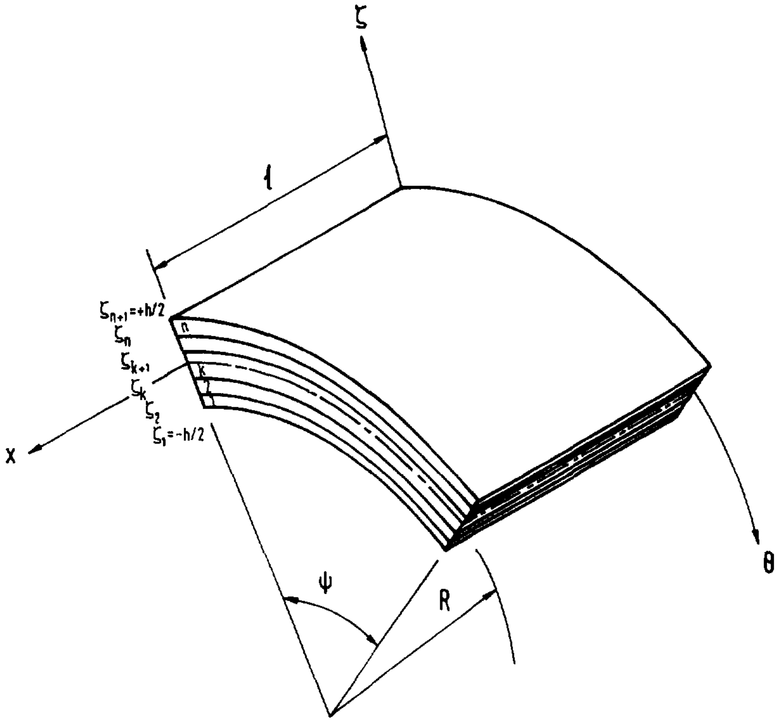

Fig. 1.

of the variables, u and v. A theory accounting for the nonlinear variation of the in-plane displacement field through the thickness of the laminated has been recently proposed by Di Sciuva (1992). The geometric and stress continuity conditions at the interfaces and the static condition of zero transverse shear stresses at the free surfaces were fulfilled *a priori*, at least for symmetric laminates. In a separate development, another theory accounting for interlaminae continuity conditions was put forward earlier by Librescu and Schmidt (1990) and applied to anisotropic composite laminated shells.

It is the objective of this paper to apply a variationally consistent simple higher-order layerwise theory to the vibration of orthotropic shells. The model accounts for a cubic variation in both the in-plane displacements and transverse shear stresses with the latter giving zero values at the top and bottom fibres without the need for shear correction factors. This theory was applied by Xavier *et al.* (1993) to the cylindrical bending of cylindrical shells showing excellent agreement with exact solutions given by Ren (1987), especially for sandwich type cross-ply shells. In this paper, the results of the present theory as applied to vibrations is compared to the solutions of the global higher-order theories of Bhimaraddi (1984) and Reddy and Liu (1985), and the simple first-order layerwise theory of Di Sciuva (1987).

THEORY

A laminated composite cylindrical shell with n layers, thickness h and length l is shown in Fig. 1. The derivation of the theory that follows is carried out such that it is applicable to any shell of arbitrary geometry. For such a shell, the assumed displacement field for each layer is given by

$$U_\alpha^k = \left(1+\frac{\zeta}{R_\alpha}\right)u_\alpha^k - \frac{\zeta}{A_\alpha}w_{,\alpha} + \left[\zeta + \frac{\left(\zeta^2 - \dfrac{R_\alpha \zeta^3}{h_0^2}\right)}{R_\alpha\left(3-\left(\dfrac{h_0}{R_\alpha}\right)^2\right)}\right]\phi_\alpha^k$$

$$U_\beta^k = \left(1+\frac{\zeta}{R_\beta}\right)u_\beta^k - \frac{\zeta}{A_\beta}w_{,\beta} + \left[\zeta + \frac{\left(\zeta^2 - \frac{R_\beta \zeta^3}{h_0^2}\right)}{R_\beta\left(3-\left(\frac{h_0}{R_\beta}\right)^2\right)}\right]\phi_\beta^k$$

$$W^k = w, \quad (k = 1, \ldots, n) \tag{1}$$

where $u_\alpha^k, u_\beta^k, w, \phi_\alpha^k, \phi_\beta^k, A_\alpha$ and A_β are functions of the α, β coordinates, the superscript k referring to the k^{th} layer. The half thickness of the shell is given by h_0 while R_α and R_β are the radii of curvatures of the reference or mid surface in the α and β coordinate directions, respectively. The surface metrics A_α and A_β serve as variable radius of curvatures that trace out the reference surface of a shell of arbitrary geometry. The displacement functions assumed enable the satisfaction of zero values of transverse shear stresses at the free surfaces *a priori* and allow for discontinous shear strains at the interfaces.

By imposing the continuity of the in-plane displacements and the transverse shear stresses at the interfaces, the following relations are obtained for a *specially* orthotropic laminate

$$\phi_\alpha^k = \mu_k \phi_\alpha^1 \tag{2}$$

$$\phi_\beta^k = \lambda_k \phi_\beta^1$$

$$u_\alpha^k = u_\alpha^1 + \sum_{j=2}^{k} (\mu_{j-1} - \mu_j)\left[\zeta_j + \frac{\left(\zeta_j^2 - \frac{R_\alpha \zeta_j^3}{h_0^2}\right)}{R_\alpha\left(3-\left(\frac{h_0}{R_\alpha}\right)^2\right)}\right]\frac{1}{\left(1+\frac{\zeta_j}{R_\alpha}\right)}\phi_\alpha^1 \tag{3}$$

$$u_\beta^k = u_\beta^1 + \sum_{j=2}^{k} (\lambda_{j-1} - \lambda_j)\left[\zeta_j + \frac{\left(\zeta_j^2 - \frac{R_\beta \zeta_j^3}{h_0^2}\right)}{R_\beta\left(3-\left(\frac{h_0}{R_\beta}\right)^2\right)}\right]\frac{1}{\left(1+\frac{\zeta_j}{R_\beta}\right)}\phi_\beta^1,$$

where $\mu_k = (Q_{55}^1/Q_{55}^k)$ and $\lambda_k = (Q_{44}^1/Q_{44}^k)$, ζ_j is the ζ-coordinate of the lower surface of the j^{th} layer and the superscript 1 refers to the first layer which has been chosen to be the reference layer for convenience. The constants Q_{44} and Q_{55}, whose values depend on fibre orientation, represent the shear moduli (modulus of rigidity) for the transverse shear stresses. The substitution of eqns (2) and (3) into eqn (1) results in an expressions for the in-plane displacements in terms of only five variables, namely $u_\alpha^1, u_\beta^1, w, \phi_\alpha^1$ and ϕ_β^1. The expressions for the transverse deflections and in-plane displacements are to be substituted into the following kinematic relations for strain

$$\varepsilon_1^k = \frac{1}{A_\alpha\left(1+\frac{\zeta}{R_\alpha}\right)}\left(U_{\alpha,\alpha}^k + \frac{U_\beta^k}{A_\beta}A_{\alpha,\beta} + \frac{A_\alpha W}{R_\alpha}\right) + \frac{1}{2}\left[\frac{1}{A_\alpha\left(1+\frac{\zeta}{R_\alpha}\right)}W_{,\alpha}\right]^2$$

$$\varepsilon_2^k = \frac{1}{A_\beta\left(1+\frac{\zeta}{R_\beta}\right)}\left(U_{\beta,\beta}^k + \frac{U_\alpha^k}{A_\alpha}A_{\beta,\alpha} + \frac{A_\beta W}{R_\beta}\right) + \frac{1}{2}\left[\frac{1}{A_\beta\left(1+\frac{\zeta}{R_\beta}\right)}W_{,\beta}\right]^2$$

$$\varepsilon_3^k = 0 \quad \varepsilon_4^k = \frac{1}{A_\beta\left(1+\dfrac{\zeta}{R_\beta}\right)}\left(W_{,\beta} + A_\beta\left(1+\frac{\zeta}{R_\beta}\right)U^k_{\beta,\zeta} - \frac{A_\beta U_\beta^k}{R_\beta}\right)$$

$$\varepsilon_5^k = \frac{1}{A_\alpha\left(1+\dfrac{\zeta}{R_\alpha}\right)}\left(W_{,\alpha} + A_\alpha\left(1+\frac{\zeta}{R_\alpha}\right)U^k_{\alpha,\zeta} - \frac{A_\alpha U_\alpha^k}{R_\alpha}\right)$$

$$\varepsilon_6^k = \frac{1}{A_\alpha\left(1+\dfrac{\zeta}{R_\alpha}\right)}\left(U^k_{\beta,\alpha} - A_{\alpha,\beta}\frac{U_\alpha^k}{A_\beta}\right) + \frac{1}{A_\beta\left(1+\dfrac{\zeta}{R_\beta}\right)}\left(U^k_{\alpha,\beta} - A_{\beta,\alpha}\frac{U_\beta^k}{A_\alpha}\right)$$
$$+\frac{1}{A_\alpha\left(1+\dfrac{\zeta}{R_\alpha}\right)}\frac{1}{A_\beta\left(1+\dfrac{\zeta}{R_\beta}\right)}W_{,\alpha}W_{,\beta}. \quad (4)$$

The higher-order terms in W are the Von Karman nonlinear strains that account for large deflection and are applicable only to the buckling problem. The additional nonlinear terms involving U_α and U_β given by Saada (1974) have been ignored as these displacements are considered to be small for the present problem.

With particular reference to ε_4 and ε_5 of eqns (4), it should be noted that the transverse shear strains (and hence stresses) for the present theory are cubic because the in-plane displacements U_β and U_α are cubic to start with. In the case of plates, the transverse shear stress field is normally quadratic if the displacement field is cubic due to the differentiation with respect to the thickness coordinate. It should be noted that for shells, in addition to the differentiated term, the displacement fields also appear explicitly in the strain expressions.

The general governing equations are obtained by substituting the expressions for strain and time derivatives of displacement that are in terms of the five variables, into Hamilton's principle. This gives

$$\delta u_\alpha^1: \left(A_\beta\left(N_1+\frac{M_1}{R_\alpha}\right)\right)_{,\alpha} - A_{\beta,\alpha}\left(N_2+\frac{M_2}{R_\alpha}\right) + \left(A_\alpha\left(N_{61}+\frac{M_{61}}{R_\alpha}\right)\right)_{,\beta}$$
$$+A_{\alpha,\beta}\left(N_{62}+\frac{M_{62}}{R_\alpha}\right) - A_\alpha A_\beta\left(I_1\ddot{u}_\alpha^1 - \frac{1}{A_\alpha}I_5\ddot{w}_{,\alpha} + I_4\ddot{\phi}_\alpha^1\right) = 0$$

$$\delta u_\beta^1: \left(A_\alpha\left(N_2+\frac{M_2}{R_\beta}\right)\right)_{,\beta} - A_{\alpha,\beta}\left(N_1+\frac{M_1}{R_\beta}\right) + \left(A_\beta\left(N_{62}+\frac{M_{62}}{R_\beta}\right)\right)_{,\alpha}$$
$$+A_{\beta,\alpha}\left(N_{61}+\frac{M_{61}}{R_\beta}\right) - A_\alpha A_\beta\left(I_8\ddot{u}_\beta^1 - \frac{1}{A_\beta}I_{11}\ddot{w}_{,\beta} + I_{10}\ddot{\phi}_\beta^1\right) = 0$$

$$\delta w: \left(\frac{A_\beta}{A_\alpha}M_1\right)_{,\alpha\alpha} + (M_{61}+M_{62})_{,\alpha\beta} + \left(\frac{A_\alpha}{A_\beta}M_2\right)_{,\beta\beta} - \left(\frac{1}{A_\alpha}A_{\beta,\alpha}M_2 - \frac{1}{A_\beta}A_{\alpha,\beta}M_{62}\right)_{,\alpha} \quad (5)$$
$$-\left(\frac{1}{A_\beta}A_{\alpha,\beta}M_1 - \frac{1}{A_\alpha}A_{\beta,\alpha}M_{61}\right)_{,\beta}$$
$$-A_\alpha A_\beta\left(\frac{N_1}{R_\alpha}+\frac{N_2}{R_\beta}\right) - \left(\frac{A_\beta}{A_\alpha}(A_\alpha I_5\ddot{u}_\alpha^1 - I_2\ddot{w}_{,\alpha} + A_\alpha I_7\ddot{\phi}_\alpha^1)\right)_{,\alpha}$$
$$-\left(\frac{A_\alpha}{A_\beta}(A_\beta I_{11}\ddot{u}_\beta^1 - I_2\ddot{w}_{,\beta} + A_\beta I_{12}\ddot{\phi}_\beta^1)\right)_{,\beta} - A_\alpha A_\beta I_6\ddot{w}$$

$$+\left(\frac{A_\beta}{A_\alpha}\bar{N}_1 w,_\alpha\right),_\alpha + 2\bar{N}_6 w,_{\alpha\beta} + \left(\frac{A_\alpha}{A_\beta}\bar{N}_2 w,_\beta\right),_\beta = 0$$

$$\delta\phi_\alpha^1 : (A_\beta P_{11}),_\alpha - A_\beta,_\alpha P_{21} + (A_\alpha P_{16}),_\beta + A_\alpha,_\beta P_{26} - \frac{A_\alpha A_\beta}{R_\alpha} D_1 - A_\alpha A_\beta\left(I_4 \ddot{u}_\alpha^1 - \frac{1}{A_\alpha} I_7 \ddot{w},_\alpha + I_3 \ddot{\phi}_\alpha^1\right) = 0$$

$$\delta\phi_\beta^1 : (A_\alpha P_{22}),_\beta - A_\alpha,_\beta P_{12} + (A_\beta P_{62}),_\alpha + A_\beta,_\alpha P_{61} - \frac{A_\alpha A_\beta}{R_\beta} D_2 - A_\alpha A_\beta\left(I_{10} \ddot{u}_\beta^1 - \frac{1}{A_\beta} I_{12} \ddot{w},_\beta + I_9 \ddot{\phi}_\beta^1\right) = 0.$$

The corresponding boundary conditions are given as follows. Along constant α curves,

$$u_\alpha^1 \text{ or } A_\beta\left(N_1 + \frac{M_1}{R_\alpha}\right)$$
$$u_\beta^1 \text{ or } A_\beta\left(N_{62} + \frac{M_{62}}{R_\beta}\right)$$
$$w \text{ or } \left(\frac{A_\beta}{A_\alpha} M_1\right),_\alpha + (M_{61} + M_{62}),_\beta - \left(\frac{1}{A_\alpha} A_\beta,_\alpha M_2 - \frac{1}{A_\beta} A_\alpha,_\beta M_{62}\right) - \frac{A_\beta}{A_\alpha}(A_\alpha I_5 \ddot{u}_\alpha^1 - I_2 \ddot{w},_\alpha + A_\alpha I_7 \ddot{\phi}_\alpha^1) + \frac{A_\beta}{A_\alpha}\bar{N}_1 w,_\alpha + \bar{N}_6 w,_\beta$$
$$w,_\alpha \text{ or } \frac{A_\beta}{A_\alpha} M_1$$
$$\phi_\alpha^1 \text{ or } A_\beta P_{11}$$
$$\phi_\beta^1 \text{ or } A_\beta P_{62}. \quad (6a)$$

Correspondingly, along constant β curves,

$$u_\alpha^1 \text{ or } A_\alpha\left(N_{61} + \frac{M_{61}}{R_\alpha}\right)$$
$$u_\beta^1 \text{ or } A_\alpha\left(N_2 + \frac{M_2}{R_\beta}\right)$$
$$w \text{ or } \left(\frac{A_\alpha}{A_\beta} M_2\right),_\beta + (M_{61} + M_{62}),_\alpha - \left(\frac{1}{A_\beta} A_\alpha,_\beta M_1 - \frac{1}{A_\alpha} A_\beta,_\alpha M_{61}\right) - \frac{A_\alpha}{A_\beta}(A_\beta I_{11} \ddot{u}_\beta^1 - I_2 \ddot{w},_\beta + A_\beta I_{12} \ddot{\phi}_\beta^1) + \frac{A_\alpha}{A_\beta}\bar{N}_2 w,_\beta + \bar{N}_6 w,_\alpha$$
$$w,_\beta \text{ or } \frac{A_\alpha}{A_\beta} M_2$$
$$\phi_\alpha^1 \text{ or } A_\alpha P_{16}$$
$$\phi_\beta^1 \text{ or } A_\alpha P_{22}. \quad (6b)$$

The stress resultants are given by

$$(N_i, M_i, D_i) = \sum_{k=1}^{n} (N_i^k, M_i^k, D_i^k)$$

$$(N_{6i}, M_{6i}) = \sum_{k=1}^{n} (N_{6i}^k, M_{6i}^k)$$

$$(P_{i6}, P_{6i}) = \sum_{k=1}^{n} (P^k_{i6}, P^k_{6i})$$

$$(P_{1i}, P_{2i}) = \sum_{k=1}^{n} (P^k_{1i}, P^k_{2i}),$$

for $i = 1, 2$ and the inertial resultants are

$$I_i = \sum_{k=1}^{n} I_i^k$$

for $i = 1, 2, \ldots 12$: while the uniform bucking load resultants are $\bar{N}_i = \sum_{k=1}^{n} \bar{N}_i^k$ for $i = 1, 2$ and 6, where

$$(N_1^k, M_1^k) = \int_{\zeta_k}^{\zeta_{k+1}} \left(1 + \frac{\zeta}{R_\beta}\right)(1, \zeta)\sigma_1^k \, \mathrm{d}\zeta$$

$$(N_2^k, M_2^k) = \int_{\zeta_k}^{\zeta_{k+1}} \left(1 + \frac{\zeta}{R_\alpha}\right)(1, \zeta)\sigma_2^k \, \mathrm{d}\zeta$$

$$(P^k_{11}, P^k_{21}) = \int_{\zeta_k}^{\zeta_{k+1}} \left[\sum_{j=2}^{k} (\mu_{j-1} - \mu_j) \left[\zeta_j + \frac{\left(\zeta_j^2 - \dfrac{R_\alpha \zeta_j^3}{h_0^2}\right)}{R_\alpha\left(3 - \left(\dfrac{h_0}{R_\alpha}\right)^2\right)} \right] \frac{\left(1 + \dfrac{\zeta}{R_\alpha}\right)}{\left(1 + \dfrac{\zeta_j}{R_\alpha}\right)} \right.$$
$$\left. + \mu_k \left[\zeta + \frac{\left(\zeta^2 - \dfrac{R_\alpha \zeta^3}{h_0^2}\right)}{R_\alpha\left(3 - \left(\dfrac{h_0}{R_\alpha}\right)^2\right)} \right] \right] \left(\left(1 + \frac{\zeta}{R_\beta}\right)\sigma_1^k, \left(1 + \frac{\zeta}{R_\alpha}\right)\sigma_2^k\right) \mathrm{d}\zeta$$

$$(P^k_{12}, P^k_{22}) = \int_{\zeta_k}^{\zeta_{k+1}} \left[\sum_{j=2}^{k} (\lambda_{j-1} - \lambda_j) \left[\zeta_j + \frac{\left(\zeta_j^2 - \dfrac{R_\beta \zeta_j^3}{h_0^2}\right)}{R_\beta\left(3 - \left(\dfrac{h_0}{R_\beta}\right)^2\right)} \right] \frac{\left(1 + \dfrac{\zeta}{R_\beta}\right)}{\left(1 + \dfrac{\zeta_j}{R_\beta}\right)} \right.$$
$$\left. + \lambda_k \left[\zeta + \frac{\left(\zeta^2 - \dfrac{R_\beta \zeta^3}{h_0^2}\right)}{R_\beta\left(3 - \left(\dfrac{h_0}{R_\beta}\right)^2\right)} \right] \right] \left(\left(1 + \frac{\zeta}{R_\beta}\right)\sigma_1^k, \left(1 + \frac{\zeta}{R_\alpha}\right)\sigma_2^k\right) \mathrm{d}\zeta$$

$$(N^k_{61}, M^k_{61}) = \int_{\zeta_k}^{\zeta_{k+1}} \left(1 + \frac{\zeta}{R_\alpha}\right)(1, \zeta)\sigma_6^k \, \mathrm{d}\zeta$$

$$(P^k_{16}, P^k_{26}) = \int_{\zeta_k}^{\zeta_{k+1}} \left[\sum_{j=2}^{k} (\mu_{j-1} - \mu_j) \left[\zeta_j + \frac{\left(\zeta_j^2 - \dfrac{R_\alpha \zeta_j^3}{h_0^2}\right)}{R_\alpha\left(3 - \left(\dfrac{h_0}{R_\alpha}\right)^2\right)} \right] \frac{\left(1 + \dfrac{\zeta}{R_\alpha}\right)}{\left(1 + \dfrac{\zeta_j}{R_\alpha}\right)} \right.$$

$$+\mu_k\left[\zeta+\frac{\left(\zeta^2-\dfrac{R_\alpha\zeta^3}{h_0^2}\right)}{R_\alpha\left(3-\left(\dfrac{h_0}{R_\alpha}\right)^2\right)}\right]\Bigg]\left(\left(1+\frac{\zeta}{R_\alpha}\right)\sigma_6^k,\left(1+\frac{\zeta}{R_\beta}\right)\sigma_6^k\right)\mathrm{d}\zeta$$

$$(N_{62}^k, M_{62}^k)=\int_{\zeta_k}^{\zeta_{k+1}}\left(1+\frac{\zeta}{R_\beta}\right)(1,\zeta)\sigma_6^k\,\mathrm{d}\zeta$$

$$(P_{61}^k,P_{62}^k)=\int_{\zeta_k}^{\zeta_{k+1}}\Bigg[\sum_{j=2}^{k}(\lambda_{j-1}-\lambda_j)\left[\zeta_j+\frac{\left(\zeta_j^2-\dfrac{R_\beta\zeta_j^3}{h_0^2}\right)}{R_\beta\left(3-\left(\dfrac{h_0}{R_\beta}\right)^2\right)}\right]\frac{\left(1+\dfrac{\zeta}{R_\beta}\right)}{\left(1+\dfrac{\zeta_j}{R_\beta}\right)}$$

$$+\lambda_k\left[\zeta+\frac{\left(\zeta^2-\dfrac{R_\beta\zeta^3}{h_0^2}\right)}{R_\beta\left(3-\left(\dfrac{h_0}{R_\beta}\right)^2\right)}\right]\Bigg]\left(\left(1+\frac{\zeta}{R_\alpha}\right)\sigma_6^k,\left(1+\frac{\zeta}{R_\beta}\right)\sigma_6^k\right)\mathrm{d}\zeta$$

$$D_1^k=\int_{\zeta_k}^{\zeta_{k+1}}\mu_k\left[R_\alpha+\frac{\left(2R_\alpha\left(\zeta-\dfrac{\zeta^3}{h_0^2}\right)+\zeta^2\left(1-3\left(\dfrac{R_\alpha}{h_0}\right)^2\right)\right)}{R_\alpha\left(3-\left(\dfrac{h_0}{R_\alpha}\right)^2\right)}\right]\left(1+\frac{\zeta}{R_\beta}\right)\sigma_5^k\,\mathrm{d}\zeta$$

$$D_2^k=\int_{\zeta_k}^{\zeta_{k+1}}\lambda_k\left[R_\beta+\frac{\left(2R_\beta\left(\zeta-\dfrac{\zeta^3}{h_0^2}\right)+\zeta^2\left(1-3\left(\dfrac{R_\beta}{h_0}\right)^2\right)\right)}{R_\beta\left(3-\left(\dfrac{h_0}{R_\beta}\right)^2\right)}\right]\left(1+\frac{\zeta}{R_\alpha}\right)\sigma_4^k\,\mathrm{d}\zeta$$

$$\bar{N}_1^k=\int_{\zeta_k}^{\zeta_{k+1}}\frac{1}{\left(1+\dfrac{\zeta}{R_\alpha}\right)}\left(1+\frac{\zeta}{R_\beta}\right)\sigma_1^k\,\mathrm{d}\zeta$$

$$\bar{N}_2^k=\int_{\zeta_k}^{\zeta_{k+1}}\frac{1}{\left(1+\dfrac{\zeta}{R_\beta}\right)}\left(1+\frac{\zeta}{R_\alpha}\right)\sigma_2^k\,\mathrm{d}\zeta$$

$$\bar{N}_6^k=\int_{\zeta_k}^{\zeta_{k+1}}\sigma_6^k\,\mathrm{d}\zeta \tag{7}$$

$$I_1^k=\int_{\zeta_k}^{\zeta_{k+1}}\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)^3\left(1+\frac{\zeta}{R_\beta}\right)\mathrm{d}\zeta$$

$$I_2^k=\int_{\zeta_k}^{\zeta_{k+1}}\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)\zeta^2\,\mathrm{d}\zeta$$

$$I_3^k=\int_{\zeta_k}^{\zeta_{k+1}}\Bigg[\sum_{j=2}^{k}(\mu_{j-1}-\mu_j)\left[\zeta_j+\frac{\left(\zeta_j^2-\dfrac{R_\alpha\zeta_j^3}{h_0^2}\right)}{R_\alpha\left(3-\left(\dfrac{h_0}{R_\alpha}\right)^2\right)}\right]\frac{\left(1+\dfrac{\zeta}{R_\alpha}\right)}{\left(1+\dfrac{\zeta_j}{R_\alpha}\right)}$$

$$+\mu_k\left[\zeta+\frac{\left(\zeta^2-\dfrac{R_\alpha\zeta^3}{h_0^2}\right)}{R_\alpha\left(3-\left(\dfrac{h_0}{R_\alpha}\right)^2\right)}\right]\Bigg]^2\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)\mathrm{d}\zeta$$

$$I_4^k=\int_{\zeta_k}^{\zeta_{k+1}}\Bigg[\sum_{j=2}^{k}(\mu_{j-1}-\mu_j)\left[\zeta_j+\frac{\left(\zeta_j^2-\dfrac{R_\alpha\zeta_j^3}{h_0^2}\right)}{R_\alpha\left(3-\left(\dfrac{h_0}{R_\alpha}\right)^2\right)}\right]\frac{\left(1+\dfrac{\zeta}{R_\alpha}\right)}{\left(1+\dfrac{\zeta_j}{R_\alpha}\right)}$$

$$+\mu_k\left[\zeta+\frac{\left(\zeta^2-\dfrac{R_\alpha\zeta^3}{h_0^2}\right)}{R_\alpha\left(3-\left(\dfrac{h_0}{R_\alpha}\right)^2\right)}\right]\Bigg]\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)^2\left(1+\frac{\zeta}{R_\beta}\right)\mathrm{d}\zeta$$

$$I_5^k=\int_{\zeta_k}^{\zeta_{k+1}}\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)^2\left(1+\frac{\zeta}{R_\beta}\right)\zeta\,\mathrm{d}\zeta$$

$$I_6^k=\int_{\zeta_k}^{\zeta_{k+1}}\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)\mathrm{d}\zeta$$

$$I_7^k=\int_{\zeta_k}^{\zeta_{k+1}}\Bigg[\sum_{j=2}^{k}(\mu_{j-1}-\mu_j)\left[\zeta_j+\frac{\left(\zeta_j^2-\dfrac{R_\alpha\zeta_j^3}{h_0^2}\right)}{R_\alpha\left(3-\left(\dfrac{h_0}{R_\alpha}\right)^2\right)}\right]\frac{\left(1+\dfrac{\zeta}{R_\alpha}\right)}{\left(1+\dfrac{\zeta_j}{R_\alpha}\right)}$$

$$+\mu_k\left[\zeta+\frac{\left(\zeta^2-\dfrac{R_\alpha\zeta^3}{h_0^2}\right)}{R_\alpha\left(3-\left(\dfrac{h_0}{R_\alpha}\right)^2\right)}\right]\Bigg]\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)\zeta\,\mathrm{d}\zeta$$

$$I_8^k=\int_{\zeta_k}^{\zeta_{k+1}}\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)^3\mathrm{d}\zeta$$

$$I_9^k=\int_{\zeta_k}^{\zeta_{k+1}}\Bigg[\sum_{j=2}^{k}(\lambda_{j-1}-\lambda_j)\left[\zeta_j+\frac{\left(\zeta_j^2-\dfrac{R_\beta\zeta_j^3}{h_0^2}\right)}{R_\beta\left(3-\left(\dfrac{h_0}{R_\beta}\right)^2\right)}\right]\frac{\left(1+\dfrac{\zeta}{R_\beta}\right)}{\left(1+\dfrac{\zeta_j}{R_\beta}\right)}$$

$$+\lambda_k\left[\zeta+\frac{\left(\zeta^2-\dfrac{R_\beta\zeta^3}{h_0^2}\right)}{R_\beta\left(3-\left(\dfrac{h_0}{R_\beta}\right)^2\right)}\right]\Bigg]^2\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)\mathrm{d}\zeta$$

$$I_{10}^k=\int_{\zeta_k}^{\zeta_{k+1}}\Bigg[\sum_{j=2}^{k}(\lambda_{j-1}-\lambda_j)\left[\zeta_j+\frac{\left(\zeta_j^2-\dfrac{R_\beta\zeta_j^3}{h_0^2}\right)}{R_\beta\left(3-\left(\dfrac{h_0}{R_\beta}\right)^2\right)}\right]\frac{\left(1+\dfrac{\zeta}{R_\beta}\right)}{\left(1+\dfrac{\zeta_j}{R_\beta}\right)}$$

$$+\lambda_k\left[\zeta+\frac{\left(\zeta^2-\frac{R_\beta\zeta^3}{h_0^2}\right)}{R_\beta\left(3-\left(\frac{h_0}{R_\beta}\right)^2\right)}\right]\Bigg]\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)^2\mathrm{d}\zeta$$

$$I_{11}^k=\int_{\zeta_k}^{\zeta_{k+1}}\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)^2\zeta\,\mathrm{d}\zeta$$

$$I_{12}^k=\int_{\zeta_k}^{\zeta_{k+1}}\Bigg[\sum_{j=2}^{k}(\lambda_{j-1}-\lambda_j)\left[\zeta_j+\frac{\left(\zeta_j^2-\frac{R_\beta\zeta_j^3}{h_0^2}\right)}{R_\beta\left(3-\left(\frac{h_0}{R_\beta}\right)^2\right)}\right]\frac{\left(1+\frac{\zeta}{R_\beta}\right)}{\left(1+\frac{\zeta_j}{R_\beta}\right)}$$

$$+\lambda_k\left[\zeta+\frac{\left(\zeta^2-\frac{R_\beta\zeta^3}{h_0^2}\right)}{R_\beta\left(3-\left(\frac{h_0}{R_\beta}\right)^2\right)}\right]\Bigg]\rho_k\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)\zeta\,\mathrm{d}\zeta$$

of which the terms under the summation sign are zero for $k = 1$. The usual constitutive equations with plane-stress reduced elastic constants will be used for each orthotropic layer of the laminate considered in the present analysis.

The buckling loads $\bar{N}_1$, $\bar{N}_2$ and $\bar{N}_3$ are the forces required to keep the shell in equilibrium under the prevailing conditions. The equations (5) are linear despite the nonlinear deformation (owing to large deflection W) of the shell due to the buckling load definitions given in eqns (7). In order to obtain solutions, these loads are normally considered as eigenvalues and are solved for in very much the same way as free vibration problems.

BUCKLING AND VIBRATION OF CYLINDRICAL SHELL PANELS

The theory outlined in this paper is applied to the buckling and vibration of specially orthotropic cylindrical shell panels subjected to simply-supported boundary conditions. For this particular case, the α-coordinate reduces to the x-coordinate which has been chosen to run along the axis of the cylindrical shell, the β-coordinate is now the θ-coordinate which traces a circumferential path while the ζ-coordinate being linear is left unchanged and is orthogonal to x and θ. It should be noted that along the x-axis, the radius of curvature is infinately large and hence the displacement and strain expressions in this direction reduce to that given by the theory of Lee *et al.* (1990). The numerical results are presented for (unless otherwise stated) shells of dimensions, $R = 5$, $\psi = \pi/3$ and $l = 25$, the material constants for each orthotropic layer are taken to be related as follows: $E_L = 25E_T$, $G_{LT} = 0.5E_T$, $G_{TT} = 0.2E_T$ and $\nu_{LT} = 0.25$, where L and T denote the directions along and perpendicular to the fibre, respectively. For the buckling problem in particular, the in-plane and shear loadings, $\bar{N}_2$ and $\bar{N}_6$, respectively, are taken to be zero for simplicity. The simply-supported (free in the in-plane normal directions) boundary conditions of the cylindrical shell panel are satisfied by the following functions:

$$u_x = u_{x0}\cos\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi\theta}{\psi}\right)$$

$$u_\theta = u_{\theta 0}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi\theta}{\psi}\right)$$

$$
\begin{aligned}
w &= w_0 \sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi\theta}{\psi}\right) \\
\phi_x &= \phi_{x0} \cos\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi\theta}{\psi}\right) \\
\phi_\theta &= \phi_{\theta 0} \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi\theta}{\psi}\right).
\end{aligned} \tag{8}
$$

where the coefficients of the functions are constants to solve for. The following three types of shell panel arrangements are considered:

(1) one-layer shells with fibres plied in the θ direction;
(2) two-layer shells with layers of equal thickness and fibres parallel to the θ and x directions in the bottom and top layers, respectively;
(3) three-layer shells of equal thickness with fibres plied in the θ and x directions in the outer and central layers, respectively.

The critical buckling load, natural frequency and stresses are non-dimensionalized as follows together with the additional definition of a performance gauging index;

$$
(\bar{\sigma}_{xx}, \bar{\sigma}_{\theta\theta}, \bar{\tau}_{x\theta}) = \frac{(\sigma_{xx}, \sigma_{\theta\theta}, \tau_{x\theta})}{E_L} hS^2 \quad (\bar{\tau}_{x\zeta}, \bar{\tau}_{\theta\zeta}) = \frac{(\tau_{x\zeta}, \tau_{\theta\zeta})}{E_L} hS^3
$$

$$
\bar{\omega} = \omega l^2 \left(\frac{\rho}{E_L h^2 S}\right)^{1/2} \quad \mathrm{GPI} = \frac{|\sigma|}{\rho h \omega^2 S^{3/2}} \quad \bar{N} = \frac{\bar{N}_1}{E_T h} \quad S = \frac{R}{h}
$$

where GPI refers to the General Performance Index, $|\sigma|$ is the magnitude of bending stress within the bottom fibre of the shell in the circumferential direction, normally the largest of all the components of stress for the cases considered, and ω is the natural frequency.

The GPI is a measure of the natural frequency and stress predicting capability of a theory. From a conservative standpoint, lower natural frequencies and larger stresses indicate a more accurate model and, therefore, the GPI has been non-dimensionalized such that it would be indicative of such cases. It should be noted that the GPI is a relative measure and not an absolute one.

The in-plane bending stress has been chosen for the definition of the GPI instead the out-of-plane transverse shear stress on account of the fact that unlike bending stresses, various shell theories give maximum values of transverse shear stresses at different locations along the thickness of the laminate. In order to evaluate the performances of shell theories at fixed locations within the laminate, the in-plane bending stress (value largest at the free-surface) has been chosen.

NUMERICAL RESULTS

The numerical results are presented in Tables 1–10. The natural frequencies and the General Performance Index (GPI) for (90°) orthotropic shells are indicated in Table 1. In what follows, the higher-order shear deformation theory by Bhimaraddi (1984), the simplified higher-order theory of Reddy and Liu (1985) and Di Sciuva's (1987) shear deformation theory will be denoted as HSDT, SHOT and DST, respectively, for convenience. Over a large range of radius-to-thickness ratios S, the present theory predicts generally lower values of natural frequency, $\bar{\omega}$ although the GPI indicates better overall performance by the HSDT. Considering the higher circumferential modes ($m = 2$, $n = 1$ and $m = 2$, $n = 2$), the natural frequencies as predicted by the present theory at $S = 4$, are significantly smaller than those of the other theories. The corresponding results for (90°/0°) laminated

Table 1. Natural frequencies and general performance index of (90°) cylindrical orthotropic shell under sinusoidal loading

Model	S	$m=1, n=1$ $\bar{\omega}$	GPI	$m=1, n=2$ $\bar{\omega}$	GPI	$m=2, n=1$ $\bar{\omega}$	GPI	$m=2, n=2$ $\bar{\omega}$	GPI
Bhimaraddi		14.32	3.11	14.44	3.05	38.59	0.96	38.56	0.96
Reddy and Liu		14.18	2.56	14.32	2.51	39.57	0.70	39.57	0.70
Di Sciuva	4	14.92	1.95	15.03	1.92	38.48	0.40	38.45	0.40
Present		14.21	3.00	14.33	2.94	37.90	0.91	37.88	0.91
		14.40	3.05	14.61	2.95	47.28	0.80	47.32	0.80
		14.37	2.92	14.59	2.83	47.35	0.74	47.40	0.74
	10	14.76	2.81	14.98	2.73	49.61	0.60	49.65	0.60
		14.39	3.04	14.61	2.94	47.23	0.79	47.27	0.79
		11.65	3.90	12.10	3.61	46.22	0.88	46.28	0.88
		11.65	3.83	12.10	3.55	46.20	0.86	46.26	0.86
	20	11.75	3.83	12.19	3.55	47.42	0.82	47.48	0.82
		11.65	3.90	12.10	3.61	46.21	0.88	46.27	0.88
		7.82	5.79	9.22	4.12	34.06	1.28	34.13	1.27
		7.82	5.75	9.23	4.09	34.06	1.27	34.13	1.26
	50	7.83	5.77	9.23	4.12	34.25	1.26	34.33	1.26
		7.82	5.79	9.22	4.12	34.06	1.28	34.13	1.27
		5.83	7.40	9.00	3.04	24.74	1.78	24.85	1.76
		5.83	7.37	9.01	3.03	24.73	1.77	24.85	1.76
	100	5.83	7.39	9.01	3.04	24.77	1.77	24.89	1.76
		5.83	7.40	9.00	3.04	24.74	1.78	24.85	1.76
		5.09	4.30	27.33	0.09	11.45	3.74	15.42	2.04
		5.09	4.30	27.33	0.09	11.45	3.74	15.42	2.04
	500	5.09	4.30	27.33	0.09	11.45	3.74	15.42	2.04
		5.09	4.30	27.33	0.09	11.45	3.74	15.42	2.04

Table 2. Natural frequencies and General Performance Index of (90°/0°) cylindrical laminated shell under sinusoidal loading

Model	S	$m=1, n=1$ $\bar{\omega}$	GPI	$m=1, n=2$ $\bar{\omega}$	GPI	$m=2, n=1$ $\bar{\omega}$	GPI	$m=2, n=2$ $\bar{\omega}$	GPI
Bhimaraddi		10.37	5.67	11.15	4.87	32.06	1.54	32.33	1.51
Reddy and Liu		9.96	5.17	10.76	4.40	30.95	1.46	31.26	1.43
Di Sciuva	4	11.08	4.96	11.82	4.33	36.46	1.28	36.67	1.26
Present		11.13	5.50	11.85	4.82	37.89	1.64	38.10	1.62
		7.76	7.17	8.97	5.28	29.79	1.82	30.10	1.78
		7.70	6.94	8.92	5.09	29.46	1.76	29.77	1.72
	10	7.88	7.03	9.08	5.22	31.47	1.68	31.76	1.65
		7.87	7.16	9.06	5.32	31.40	1.79	31.69	1.76
		5.94	8.70	8.14	4.47	23.54	2.32	24.87	2.25
		5.93	8.56	8.13	4.39	23.46	2.28	24.79	2.21
	20	5.97	8.66	8.16	4.47	23.96	2.27	24.28	2.21
		5.96	8.70	8.15	4.49	23.92	2.31	24.24	2.24
		5.05	7.50	9.74	1.83	15.42	3.53	15.99	3.27
		5.04	7.45	9.73	1.81	15.40	3.51	15.98	3.25
	50	5.05	7.50	9.74	1.83	15.46	3.52	16.03	3.26
		5.05	7.50	9.74	1.83	15.46	3.53	16.03	3.27
		5.71	4.05	12.97	0.64	11.04	4.87	12.27	3.91
		5.71	4.04	12.97	0.63	11.04	4.85	12.27	3.90
	100	5.71	4.05	12.97	0.64	11.05	4.86	12.28	3.91
		5.71	4.05	12.97	0.64	11.05	4.87	12.28	3.91
		11.51	0.39	28.32	0.01	6.26	6.74	12.94	1.52
		11.51	0.39	28.32	0.01	6.26	6.74	12.94	1.52
	500	11.51	0.39	28.32	0.01	6.26	6.74	12.94	1.52
		11.51	0.39	28.32	0.01	6.26	6.74	12.94	1.52

shells are shown in Table 2. For the cross-ply arrangement considered, the natural frequencies, $\bar{\omega}$ of the SHOT, even for higher modes, are observed to be the lowest of all the theories compared, despite the simplifications made as previously mentioned. However, the GPI of the SHOT indicate that the stress predicting capability of the other models are better by comparison. At $S = 4$, HSDT indicates the largest GPI for the lowest circumferential modes ($m = 1, n = 1$ and $m = 1, n = 2$). For the same radius-to-thickness ratio, the higher modes

Table 3. Natural frequencies and General Performance Index of (90°/0°/90°) cylindrical laminated shell under sinusoidal loading

Model	S	$m = 1, n = 1$ $\bar{\omega}$	GPI	$m = 1, n = 2$ $\bar{\omega}$	GPI	$m = 2, n = 1$ $\bar{\omega}$	GPI	$m = 2, n = 2$ $\bar{\omega}$	GPI
Bhimaraddi		12.39	3.53	12.77	3.29	32.57	1.13	32.68	1.12
Reddy and Liu		12.25	2.96	12.65	2.74	33.57	0.83	33.70	0.82
Di Sciuva	4	12.10	2.98	12.50	2.76	31.90	1.14	32.01	1.13
Present		12.06	3.93	12.45	3.66	34.88	1.58	34.98	1.57
		13.36	3.21	13.95	2.91	41.15	0.85	41.25	0.84
		13.33	3.09	13.93	2.80	41.25	0.80	41.36	0.79
	10	13.25	3.10	13.84	2.81	40.60	0.81	40.70	0.80
		12.99	3.35	13.59	3.03	39.85	1.00	39.96	1.00
		11.37	3.95	12.53	3.21	42.60	0.94	42.72	0.94
		11.37	3.90	12.53	3.16	42.58	0.93	42.71	0.92
	20	11.34	3.92	12.50	3.18	42.26	0.91	42.38	0.91
		11.25	4.00	12.41	3.24	41.21	0.99	41.53	0.98
		8.22	5.18	11.48	2.56	32.98	1.33	33.19	1.31
		8.22	5.15	11.49	2.55	32.97	1.32	33.19	1.31
	50	8.22	5.18	11.48	2.56	32.91	1.33	33.13	1.31
		8.20	5.19	11.47	2.56	32.72	1.34	32.94	1.32
		7.11	4.92	13.33	1.30	24.24	1.84	24.71	1.77
		7.11	4.90	13.33	1.30	24.24	1.83	24.71	1.76
	100	7.11	4.91	13.33	1.30	24.22	1.84	24.70	1.76
		7.11	4.92	13.33	1.30	24.19	1.84	24.66	1.77
		10.68	0.94	27.33	0.09	11.45	3.74	15.42	2.04
		10.68	0.94	27.33	0.09	11.45	3.74	15.42	2.04
	500	10.68	0.94	27.33	0.09	11.45	3.74	15.42	2.04
		10.68	0.94	27.33	0.09	11.45	3.74	15.42	2.04

($m = 2, n = 1$ and $m = 2, n = 2$) show that the present theory generally performs best. The results to the natural frequencies, $\bar{\omega}$ and general performance, GPI of the theories for (90°/0°/90°) laminated, cross-ply shells are given in Table 3. At $S = 4$, the present theory indicates the highest values of the GPI for all modes, although the DST predicts the lowest values of natural frequencies for the higher modes ($m = 2$, $n = 1$ and $m = 2$, $n = 2$), indicating that despite predicting higher values of natural frequencies at such modes, the present theory predicts better circumferential bending stress thereby enhancing its GPI magnitude.

The bending and shear stresses induced at the maximum amplitude of vibration, for (90°), (90°/0°) and (90°/0°/90°) shells, respectively, are shown in Tables 4–6. The results to the shear stresses are not discussed, as the maximum values as predicted by the various theories do not occur at the same point along the thickness of the laminate unlike the bending stresses. For computational convenience, the shear stresses were evaluated at points at which all the models appeared to generally predict large values. Table 4 shows that the HSDT predicts the largest values of circumferential bending stress, $\bar{\sigma}_{\theta\theta}(l/2, \psi/2, \mp h/2)$, although Tables 5 and 6 indicate that it is the present theory that predicts the largest values for the (90°/0°) and (90°/0°/90°) arrangements. Table 6 also shows that it is not sufficient to use a first-order layerwise theory for thick shells as the HSDT and SHOT predict larger values of bending stress as compared to the DST. Comparing the HSDT and SHOT, Tables 4–6 indicate that the simplifications of the SHOT would have a negligible effect on natural frequency and stress prediction only at radius-to-thickness ratios of 20 or greater. At $S = 4$, however, the simplifications result in the reduction of predicted circumferential bending stresses, $\bar{\sigma}_{\theta\theta}(l/2, \psi/2, -h/2)$ by approximately 24, 19 and 22% for single-layer, two-layer and three-layer shells, respectively. Based on Tables 1–3, the natural frequencies, $\bar{\omega}$ are, however, favourably smaller by 4% for the two-layer shell while the corresponding figures for single-layer and three-layer shells are a mere 1%.

The variations of natural frequencies and the GPI with the angle ψ subtended by the ends of the shell at $S = 4$, are presented in Table 7. The present theory predicts the lowest values of natural frequencies and the largest GPI over the whole range of shell depth considered. Table 8 shows the variation of the natural frequencies and General Performance Index with base layer-to-core modulus of elasticity ratios. With the exception of the values

Table 4. Non-dimensionalized stresses in (90°) cylindrical orthotropic shell under sinusoidal loading

Model	S	$\bar{\sigma}_{xx}$ $(l/2, \psi/2, \mp h/2)$	$\bar{\sigma}_{\theta\theta}$ $(l/2, \psi/2, \mp h/2)$	$\bar{\tau}_{x\theta}$ $(0, 0, \mp h/2)$	$\bar{\tau}_{xz}$ $(0, \psi/2, 0)$	$\bar{\tau}_{\theta z}$ $(l/2, 0, 0)$
Bhimaraddi		−0.01585	−2.0422	0.01150	0.00884	0.7155
		0.03122	1.1918	−0.02925		
Reddy and Liu		−0.01227	−1.6525	0.01325	0.01156	0.7169
	4	0.03340	1.4855	−0.03310		
Di Sciuva		−0.00853	−1.3948	0.00693	0.00920	0.8859
		0.03023	1.0100	−0.03015		
Present		−0.01496	−1.9401	0.01149	0.00893	0.7034
		0.03174	1.2579	−0.02925		
		−0.01999	−3.1973	0.02406	0.02289	2.0306
		0.05513	2.7613	−0.03201		
		−0.01852	−3.0506	0.02560	0.02451	2.0267
	10	0.05631	2.8828	−0.03493		
		−0.01878	−3.0993	0.02311	0.02367	2.1774
		0.05501	2.7283	−0.03235		
		−0.01984	−3.1814	0.02400	0.02288	2.0262
		0.05524	2.7733	−0.03206		
		−0.01152	−3.7930	0.03409	0.02949	2.6907
		0.07783	3.5517	−0.02907		
		−0.01078	−3.7221	0.03574	0.03096	2.6889
	20	0.07848	3.6146	−0.03143		
		−0.01136	−3.7814	0.03396	0.02984	2.7494
		0.07785	3.5498	−0.02915		
		−0.01149	−3.7905	0.03409	0.02950	2.6876
		0.07785	3.5538	−0.02908		
		0.03367	−4.0000	0.04809	0.03254	2.9576
		0.12875	3.9048	−0.01789		
		0.03396	−3.9725	0.04986	0.03404	2.9475
		0.12903	3.9307	−0.01995		
	50	0.03367	−4.0000	0.04808	0.03237	2.9619
		0.12875	3.9049	−0.01790		
		0.03367	−4.0000	0.04809	0.03214	2.9552
		0.12877	3.9049	−0.01789		
		0.11392	−4.0161	0.06606	0.03151	2.9982
		0.20992	3.9777	−0.00036		
		0.11406	−4.0024	0.06790	0.03594	2.9992
	100	0.21005	3.9907	−0.00234		
		0.11392	−4.0161	0.06606	0.03014	3.0123
		0.20992	3.9777	−0.00036		
		0.11392	−4.0161	0.06606	0.03200	2.9976
		0.20992	3.9777	−0.00036		
		0.76064	−3.9848	0.20181	0.03265	3.0136
		0.85694	4.0381	0.13524		
		0.76067	−3.9819	0.20370	0.03407	3.0138
	500	0.85697	4.0406	0.13333		
		0.76064	−3.9848	0.20181	0.03265	3.0138
		0.85694	4.0381	0.13524		
		0.76064	−3.9848	0.20181	0.03265	3.0107
		0.85694	4.0381	0.13524		

at $E_L/E_T = 2$, at $S = 4$, the present theory predicts the lowest values of the natural frequency over the entire range. The GPI clearly indicates that the present model generally performs the best over the range base layer-to-core modulus ratios, E_L/E_T and radius-to-thickness ratios, S.

Results to the buckling of shells at $S = 4.0$ are presented in Tables 9 and 10 indicate that the present theory generally predicts conservatively smaller critical buckling loads, $\bar{N}$ for symmetric shell configurations. However, for the antisymmetric setup, the SHOT clearly gives the most favourable results, followed by that of the HSDT. The layerwise theories do not perform well in this case.

CONCLUSIONS

A performance gauging factor called the General Performance Index has been derived to enable a quick assessment of the dynamic performance of shell theories. The index has

Table 5. Non-dimensionalized stresses in (90°/0°) cylindrical laminated shell under sinusoidal loading

Model	S	$\bar{\sigma}_{xx}$ $(l/2, \psi/2, \mp h/2)$	$\bar{\sigma}_{\theta\theta}$ $(l/2, \psi/2, \mp h/2)$	$\bar{\tau}_{x\theta}$ $(0, 0, \mp h/2)$	$\bar{\tau}_{x\zeta}$ $(0, \psi/2, -h/4)$	$\bar{\tau}_{\theta\zeta}$ $(l/2, 0, h/4)$
Bhimaraddi		−0.02566	−1.9524	0.03997	0.02397	0.0774
		0.21011	0.1244	−0.01266		
Reddy and Liu		−0.02254	−1.6429	0.03760	0.02490	0.0863
	4	0.21602	0.1451	−0.01516		
Di Sciuva		−0.02517	−1.9502	0.03868	0.02521	0.0808
		0.20464	0.1152	−0.01133		
Present		−0.27402	−2.1808	0.03931	0.02459	0.0760
		0.21052	0.1192	−0.01220		
		−0.02516	−2.1829	0.05806	0.03963	0.1328
		0.34820	0.2091	−0.00748		
		−0.02409	−2.0847	0.05749	0.03936	0.1370
	10	0.35289	0.2188	−0.00896		
		−0.02534	−2.2107	0.05793	0.03986	0.1349
		0.34713	0.2086	−0.00734		
		−0.02565	−2.2452	0.05797	0.03962	0.1332
		0.34820	0.2091	−0.00748		
		−0.01766	−2.1978	0.08109	0.05592	0.1437
		0.55546	0.2312	0.01311		
		−0.01712	−2.1531	0.08117	0.05596	0.1461
	20	0.55915	0.2364	0.01789		
		−0.01772	−2.2067	0.08104	0.05598	0.1443
		0.55523	0.2311	0.01314		
		−0.01778	−2.2147	0.08103	0.05589	0.1439
		0.55549	0.2312	0.01311		
		0.00700	−2.1584	0.14777	0.10207	0.1425
		1.16832	0.2448	0.07924		
		0.00727	−2.1415	0.14824	0.10234	0.1436
		1.17133	0.2470	0.07800		
	50	0.00700	−2.1600	0.14774	0.10213	0.1426
		1.16831	0.2448	0.07925		
		0.00699	−2.1611	0.14774	0.10211	0.1425
		1.16834	0.2448	0.07925		
		0.04820	−2.1144	0.25865	0.17894	0.1354
		2.18847	0.2573	0.19014		
		0.04839	−2.1062	0.25924	0.17922	0.1360
	100	2.19123	0.2584	0.18889		
		0.04820	−2.1148	0.25864	0.17894	0.1354
		0.20992	0.2573	0.19011		
		0.04820	−2.1151	0.25863	0.17893	0.1354
		2.18848	0.2573	0.19011		
		0.37725	−1.8556	1.14583	0.79394	0.0762
		10.34906	0.3465	1.07734		
		0.37737	−1.8541	1.14652	0.79422	0.0764
	500	10.35163	0.3467	1.07614		
		0.37725	−1.8556	1.14582	0.79393	0.0762
		10.34906	0.3465	1.07734		
		0.37725	−1.8556	1.14583	0.79392	0.0762
		10.34906	0.3465	1.07734		

been non-dimensionalized such that it would give a large value for a theory predicting large magnitudes of stresses and low values of natural frequency according to conservative natural frequency and stress prediction.

Numerical results show that while the HSDT performs comparably for (90°) orthotropic shells, the present theory clearly gives the largest values of the GPI for (90°/0°) and (90°/0°/90°) laminated shells, indicating that it would be a better choice for multi-layer shells. The SHOT predicts the lowest values of natural frequency for two-layer cross-ply shells. The simplifications of the SHOT result in a 24% reduction in the magnitude of circumferential bending stress, $\bar{\sigma}_{\theta\theta}(l/2, \psi/2, -h/2)$ when compared to the otherwise similar HSDT. The corresponding reduction for the (90°/0°) and (90°/0°/90°) arrangements are 19 and 22%, respectively. However, the simplifications favour the SHOT for natural frequencies, the reduction being 1% for one-layer and two-layer shells while the figure for a (90°/0°) laminated shell is 4%. Results also show that irrespective of the degree of

Table 6. Non-dimensionalized stresses in (90°/0°/90°) cylindrical laminated shell under sinusoidal loading

Model	S	$\bar{\sigma}_{xx}$ $(l/2, \psi/2, \mp h/2)$	$\bar{\sigma}_{\theta\theta}$ $(l/2, \psi/2, \mp h/2)$	$\bar{\tau}_{x\theta}$ $(0, 0, \mp h/2)$	$\bar{\tau}_{x\zeta}$ $(0, \psi/2, -h/3)$	$\bar{\tau}_{\theta\zeta}$ $(l/2, 0, -h/3)$
Bhimaraddi		−0.01981	−1.7342	0.02931	0.01343	0.5303
		0.02270	1.0174	−0.01428		
Reddy and Liu		−0.01668	−1.4198	0.02816	0.01309	0.4586
	4	0.02486	1.2443	−0.01769		
Di Sciuva		−0.01625	−1.3951	0.02768	0.01319	0.4967
		0.02189	0.9484	−0.01371		
Present		−0.20670	−1.8297	0.02941	0.01351	0.5560
		0.02427	1.1890	−0.01476		
		−0.02599	−2.8982	0.04516	0.02311	1.1296
		0.04356	2.4849	−0.01111		
		−0.02476	−2.7795	0.04556	0.02332	1.1018
	10	0.04452	2.5784	−0.01314		
		−0.02453	−2.7553	0.04452	0.02299	1.1135
		0.04264	2.3974	−0.01064		
		−0.02565	−2.8618	0.04507	0.02291	1.0994
		0.04336	2.4742	−0.01086		
		−0.02430	−3.6583	0.06764	0.03412	1.5033
		0.06244	3.4201	0.00376		
		−0.02369	−3.6012	0.06845	0.03447	1.4902
	20	0.06296	3.4688	0.00222		
		−0.02377	−3.6071	0.06741	0.03407	1.4964
		0.06202	3.3792	0.00396		
		−0.02395	−3.6238	0.06751	0.03397	1.4818
		0.06213	3.3917	0.00394		
		0.00157	−3.9610	0.12510	0.05969	1.6565
		0.09614	3.8910	0.05782		
		0.00181	−3.9392	0.12612	0.06012	1.6517
		0.09636	3.9114	0.05649		
	50	0.00157	−3.9518	0.12506	0.05969	1.6552
		0.09605	3.8825	0.05782		
		0.00165	−3.9530	0.12507	0.05966	1.6517
		0.09606	3.8834	0.05782		
		0.05013	−3.9783	0.21756	0.10072	1.6739
		0.14600	4.0018	0.14968		
		0.05025	−3.9677	0.21865	0.10114	1.6717
	100	0.14611	4.0123	0.14846		
		0.05015	−3.9760	0.21756	0.10074	1.6736
		0.14598	3.9996	0.14970		
		0.05015	−3.9762	0.21756	0.10072	1.6727
		0.14598	4.0000	0.14970		
		0.44275	−3.8195	0.95337	0.42814	1.6646
		0.53904	4.2028	0.88530		
		0.44277	−3.8175	0.95451	0.42862	1.6643
	500	0.53907	4.2050	0.88414		
		0.44275	−3.8195	0.95337	0.42813	1.6646
		0.53904	4.2027	0.88530		
		0.44275	−3.8195	0.95337	0.42810	1.6646
		0.53904	4.2027	0.88530		

Table 7. Natural frequencies and General Performance Index of (90°/0°/90°) cylindrical laminated shell under sinusoidal loading

Model	S	$\psi = (\pi/3)$ $\bar{\omega}$	GPI	$\psi = (\pi/2)$ $\bar{\omega}$	GPI	$\psi = (2\pi/3)$ $\bar{\omega}$	GPI	$\psi = (5\pi/6)$ $\bar{\omega}$	GPI
Bhimaraddi		12.39	3.53	6.03	7.43	3.43	11.40	2.78	6.54
Reddy and Liu		12.25	2.96	5.95	6.51	3.41	10.07	2.78	5.77
Di Sciuva	4	12.10	2.98	5.92	6.71	3.40	10.54	2.77	6.08
Present		12.06	3.93	5.81	8.03	3.36	11.73	2.77	6.44
		13.36	3.21	5.89	7.38	4.03	6.87	4.15	2.07
		13.33	3.09	5.89	7.13	4.03	6.42	4.15	2.00
	10	13.25	3.10	5.88	7.24	4.03	6.78	4.15	2.04
		12.99	3.35	5.81	7.49	4.02	6.85	4.15	2.05
		11.37	3.95	5.44	6.80	5.05	3.16	5.79	0.63
		11.37	3.90	5.44	6.70	5.05	3.12	5.79	0.62
	20	11.34	3.92	5.44	6.77	5.05	3.15	5.79	0.62
		11.25	4.00	5.42	6.81	5.05	3.15	5.79	0.63
		8.22	5.18	6.21	3.30	7.62	0.77	9.11	0.04
		8.22	5.15	6.21	3.32	7.62	0.77	9.11	0.04
	50	8.22	5.17	6.21	3.31	7.62	0.77	9.11	0.04
		8.20	5.19	6.21	3.31	7.62	0.77	9.11	0.04
		7.11	4.92	8.14	1.30	10.70	0.21	12.88	0.06
		7.11	4.90	8.14	1.30	10.70	0.21	12.88	0.06
	100	7.11	4.91	8.14	1.30	10.70	0.21	12.88	0.06
		7.11	4.92	8.14	1.30	10.70	0.21	12.88	0.06

Table 8. Natural frequencies and General Performance Index of (90°/0°/90°) cylindrical laminated shell under sinusoidal loading

Model	S	$E_L/E_T = 2$ $\bar{\omega}$	GPI	$E_L/E_T = 5$ $\bar{\omega}$	GPI	$E_L/E_T = 10$ $\bar{\omega}$	GPI	$E_L/E_T = 15$ $\bar{\omega}$	GPI
Bhimaraddi		23.34	1.91	20.40	2.33	16.95	2.65	14.87	2.94
Reddy and Liu		24.02	2.08	20.12	2.11	16.74	2.34	14.70	2.55
Di Sciuva	4	24.13	2.03	20.10	2.21	16.62	2.42	14.55	2.61
Present		23.78	2.13	19.70	2.45	16.29	2.87	14.32	3.25
		17.56	2.72	16.39	2.89	15.36	2.99	14.58	3.07
		17.54	2.63	16.36	2.80	15.32	2.90	14.54	2.97
	10	17.53	2.72	16.35	2.86	15.29	2.95	14.49	3.00
		17.47	2.74	16.23	2.92	15.10	3.05	14.26	3.16
		13.06	3.48	12.40	3.71	12.02	3.82	11.77	3.88
		13.07	3.41	12.40	3.66	12.01	3.77	11.77	3.82
	20	13.06	3.47	12.39	3.71	12.00	3.81	11.75	3.86
		13.04	3.48	12.37	3.72	11.96	3.84	11.68	3.91
		9.82	3.85	8.99	4.47	8.62	4.80	8.44	4.97
		9.82	3.81	8.99	4.44	8.62	4.77	8.44	4.94
	50	9.82	3.84	8.99	4.47	8.62	4.80	8.44	4.96
		9.18	3.85	8.99	4.47	8.61	4.80	8.43	4.97
		9.85	2.68	8.49	3.52	7.83	4.08	7.51	4.43
		9.86	2.67	8.49	3.51	7.84	4.07	7.51	4.42
	100	9.85	2.68	8.49	3.52	7.83	4.08	7.51	4.43
		9.85	2.68	8.49	3.52	7.83	4.08	7.51	4.43

Table 9. Critical buckling loads, $\bar{N}$ for the various laminate configurations of cylindrical shells under sinusoidal loading with varying modulus ratios ($h_1 = h_2 = h_3$)

Model	E_L/E_T	(90°)	(90°/0°)	(90°/0°/90°)
Bhimaraddi		1.1780	0.7537	0.8737
Reddy and Liu		1.1427	0.7239	0.8445
Di Sciuva	5	1.2768	0.8714	0.8347
Present		1.1598	0.8302	0.8323
		2.3428	1.3435	1.7368
		2.2732	1.2674	1.6807
	10	2.5395	1.5465	1.6598
		2.3066	1.5146	1.6513
		3.5069	1.8861	2.5996
		3.4032	1.7605	2.5168
	15	3.8015	2.1644	2.4848
		3.4528	2.1497	2.4702
		4.6711	2.4087	3.4626
		4.5333	2.2317	3.3530
	20	5.0676	2.7575	3.3096
		4.5989	2.7615	3.2889

Table 10. Critical buckling loads, $\bar{N}$ for the various laminate configurations of cylindrical shells under sinusoidal loading with varying thickness ratios ($E_L/E_T = 25$)

Model	h_2/h_1	(90°/0°)	(90°/0°/90°) $h_3/h_1 = 0.5$	(90°/0°/90°) $h_3/h_1 = 1.0$
Bhimaraddi		5.3546	5.5626	5.6071
Reddy and Liu		5.0330	5.4288	5.4396
Di Sciuva	0.1	6.3571	5.8032	5.7931
Present		5.2566	5.4059	5.3829
		4.0130	4.7279	4.9078
		3.6580	4.6828	4.7557
	0.5	4.9305	4.7446	4.7299
		4.5219	4.6892	4.6339
		2.9124	3.7587	4.3129
		2.6850	3.8149	4.1770
	1.0	3.3291	3.7731	4.1226
		3.3517	3.7837	4.0960
		1.7259	2.9864	3.1406
		1.6844	2.9413	3.0448
	5.0	1.7646	2.9535	3.0791
		1.7319	3.0083	3.1263

shallowness or the base layer-to-core modulus of elasticity, the present theory is generally the most suitable theory for natural frequency and stress prediction.

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